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Concentration of Stresses in the Vicinity of An Aperture in the Surface of a Circular Cylinder

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CONCENTRATION OF STRESSES IN THE VICINITY OF AN APERTURE IN THE SURFACE OF A CIRCULAR CYLINDER

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1. The equations of the theory of cylindrical shells used in the present paper are of an approximate character. They are based on the fact that the displacements u and v in the middle surface are neglected in computing the curvature and twist of the shell; the latter are expressed only by the radial component w of the displacement. Under these conditions the problem of cylindrical shells can be reduced to finding a displacement function ϕ ; if no surface loads are present, then this function ϕ is given by the differential equation of 8th order

$$(1.1) \qquad \triangle^4 \phi + \frac{12(m^2 - 1)}{m^2 a^2 h^2} \frac{\partial^4 \phi}{\partial x^4} = 0$$

Here we have used the following notation: a is the radius of the cylinder, h the thickness of the wall, m is the reciprocal of Poisson's ratio; $\triangle$ as usual stands for the Laplace operator $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$, where x is the abcissa counted along the generator of the cylinder, $y = a\theta$, θ being the azimuthal angle of the meridian plane considered.

In the following we shall denote differentiation with respect to x and y by ∂_1 and ∂_2 , respectively.

We now give the formulas expressing the displacements, forces and moments in terms of the displacement function. The axial displacement u is given by the formula

$$(1.2) \qquad u = \frac{1}{a} \partial_1 (\triangle - \frac{m+1}{m} \partial_1^2) \phi .$$

The displacement v , perpendicular to the meridian plane is found by the formula

$$v = -\frac{1}{a}\,\partial_2\left(\Delta + \frac{m+1}{m}\partial_1^{\,2}\right)\phi\,. \tag{1.3}$$

Finally for the radial displacement we have the expression

$$w = \Delta^2\phi\,. \tag{1.4}$$

The expressions for the axial, circumferential and tangential shear forces will be, respectively,

$$S_1 = \frac{Eh}{a}\partial_2^{\,2}\partial_1^{\,2}\phi\,, \quad S_3 = \frac{Eh}{a}\partial_1^{\,4}\phi\,, \quad S = -\frac{Eh}{a}\partial_2\partial_1^{\,3}\phi\,. \tag{1.5}$$

For the meridial circumferential and twisting moments G_1, G_2 and H, respectively, we have the expressions

$$G_1 = -D\left(\partial_1^{\,2} + \frac{1}{m}\partial_2^{\,2}\right)\Delta^2\phi\,, \quad G_2 = -D\left(\partial_2^{\,2} + \frac{1}{m}\partial_1^{\,2}\right)\Delta^2\phi \tag{1.6}$$

$$H = -D\,\frac{m-1}{m}\,\partial_1\partial_2\Delta^2\phi \qquad \left(D = \frac{Em^2h^3}{12(m^2-1)}\right)\,.$$

The distribution of the twisting moment H and of the shear force at the cross section x = const. is statically equivalent to the distribution of the "shear reaction"

$$Q_1^{\,*} = Q_1 + \frac{\partial H}{\partial y} = -D\partial_1\left(\partial_1^{\,2} + \frac{2m-1}{m}\partial_2^{\,2}\right)\Delta^2\phi \tag{1.7}$$

per unit of length of arc of a parallel circle x = const. In the same way the distribution of this moment and of the shear force on the cross section θ = const. is statically equivalent to the "shear reaction"

$$(1.8) \qquad Q_2^{*} = Q_2 + \frac{\partial H}{\partial x} = - D\partial_2\left(\frac{2m-1}{m}\partial_1^2 + \partial_2^2\right)\Delta^2\phi$$

per unit of length of the cylinder.

Equation (1.1) can be replaced by the system of equations

$$(1.9) \qquad \Delta^2\phi - \sqrt{\frac{12(m^2-1)}{m^2a^2h^2}}\,\frac{\partial^2\phi_1}{\partial x^2} = 0, \quad \Delta^2\phi_1 + \sqrt{\frac{12(m^2-1)}{m^2a^2h^2}}\,\frac{\partial^2\phi}{\partial x^2} = 0.$$

Let us introduce an auxiliary complex function

$$(1.10) \qquad \sigma_* = \phi + i\phi_1 \,.$$

To determine it from (1.9) we find equation

$$(1.11) \qquad \Delta^2\sigma_* + \sqrt{\frac{3(m^2-1)}{m^2}}\,\frac{2i}{ah}\,\frac{\partial^2\sigma_*}{\partial x^2} = 0.$$

From (1.9) and the shear forces and moments it follows that the forces S_1, S_2, S are expressed by imaginary parts of the function $\Delta^2\sigma_*$, whereas the bending moments, the torsion moments and the shear reactions Q_1^*, Q_2^* are expressed by the real part of the same function $\Delta^2\sigma_*$.

Therefore, introducing the function

(1.12) $$\sigma = \Delta^2 \sigma_*$$

the real part of which represents the radial displacement (this follows from (1.4)), we obtain for the moments and shear reactions, the following expressions:

(1.13)
$$S_1 = -\frac{D}{h}\sqrt{\frac{12(m^2-1)}{m^2}}\,\partial_2^{\,2}\operatorname{Im}\sigma, \quad G_1 = -D(\partial_1^{\,2} + \frac{1}{m}\partial_2^{\,2})\operatorname{Re}\sigma,$$

$$S_2 = -\frac{D}{h}\sqrt{\frac{12(m^2-1)}{m^2}}\,\partial_1^{\,2}\operatorname{Im}\sigma, \quad G_2 = -D(\partial_2^{\,2} + \frac{1}{m}\partial^2_{\,1})\operatorname{Re}\sigma,$$

$$S = \frac{D}{h}\sqrt{\frac{12(m^2-1)}{m^2}}\,\partial_1\,\partial_2\operatorname{Im}\sigma, \qquad H = -D\,\frac{m-1}{m}\,\partial_1\,\partial_2\operatorname{Re}\sigma,$$

$$Q_1^{\,*} = -D\partial_1\left(\frac{2m-1}{m}\,\partial_2^{\,2} + \partial_1^{\,2}\right)\operatorname{Re}\sigma, \quad Q_2^{\,*} = -D\partial_2\left(\frac{2m-1}{m}\partial_1^{\,2} + \partial_2^{\,2}\right)\operatorname{Re}\sigma.$$

It is clear that σ is determined from the same differential equation (1.11) as σ_*:

(1.14) $$\Delta^2\sigma + \sqrt{\frac{3(m^2-1)}{m^2}}\,\frac{2i}{ah}\,\frac{\partial^2}{\partial x^2} = 0\,.$$

Let us note that to express the forces S_1, S_2, S in terms of the imaginary part of σ we use the same formulas as are used to express the corresponding stresses in the problem of a plane stress state in terms of Airy stress function.

2. We shall now give some information concerning the solution of the differential equation (1.14). This equation can be given in the form

$$L_1 L_2 \sigma = 0, \tag{2.1}$$

where L_1 and L_2 are the differential operators

$$L_1 = \Delta + 2(1 - i)\beta\frac{\partial}{\partial x}, \quad L_2 = \Delta - 2(1 - i)\beta\frac{\partial}{\partial x} \tag{2.2}$$

with

$$\beta = \sqrt[4]{\frac{3(m^2 - 1)}{m^2}}\ \frac{1}{2\sqrt{ah}} \tag{2.3}$$

Since the operators L_1 and L_2 are commutative, the solutions of each of the equations

$$L_1\sigma = 0, \qquad L_2\sigma = 0 \tag{2.4}$$

represent a solution of equation (1.14). In the first equation of (2.4) we assume

$$\sigma = e^{-(1 - i)\beta x}\psi(x,y) \tag{2.5}$$

and in the second

$$\sigma = e^{(1 - i)\beta x}\psi(x,y) \tag{2.6}$$

Then ψ is given by equation

$$\Delta\psi + 2i\beta^2\psi = 0 \tag{2.7}$$

If ψ_1 and ψ_2 are linearly independent solutions of this equation, then the expressions

$$\psi_1 e^{-(1-i)\beta x}, \qquad \psi_2 e^{-(1-i)\beta x} \tag{2.8}$$

will be partial solutions of the first of equations (2.4), and

$$\psi_1 e^{(1-i)\beta x}, \qquad \psi_2 e^{(1-i)\beta x} \tag{2.9}$$

of the second.

Of course every linear combination of these solutions will be a solution of (1.14).

Let us introduce into the discussion Krylov's functions

$$\begin{aligned} &\Omega_1(\xi) = \cos h\xi\cos\xi, \quad \Omega_2(\xi) = \tfrac{1}{2}(\cos h\xi\sin\xi + \sin h\xi\cos\xi), \\ &\Omega_3(\xi) = \tfrac{1}{2}\sin h\xi\sin\xi, \quad \Omega_4(\xi) = \tfrac{1}{4}(\cos h\xi\sin\xi - \sin h\xi\cos\xi). \end{aligned} \tag{2.10}$$

It is easily checked that the expressions $\Omega_1(\xi) - 2i\,\Omega_3(\xi)$, $\Omega_2(\xi) - 2i\,\Omega_4(\xi)$ are linear combinations of the even and odd components of the functions $e^{-(1-i)\xi}$, $e^{(1-i)\xi}$.

We can therefore look at the solution of (1.14) in the form

$$\begin{aligned} &[\Omega_1(\beta x) - 2i\,\Omega_3(\beta x)]\psi(x,y), \\ &[\Omega_2(\beta x) - 2i\,\Omega_4(\beta x)]\psi(x,y) \end{aligned} \tag{2.11}$$

instead of (2.8) and (2.9).

In the following we shall investigate "polar" coordinates on the surface of the cylinder; i.e. we assume

$$x = \rho \cos \lambda, \qquad y = \rho \sin \lambda . \tag{2.12}$$

The curves λ = const. are the helical lines: if one unfolds the cylinder into a plane these lines go over into a system of straight lines which emit radially from the origin of coordinates.

On this plane the curves ρ = const. are circles, and on the cylinder they are curves of the same geodetic distance from the origin of the coordinates - it is into these curves that the circles go over when one bends the plane into the surface of a circular cylinder.

The transformation (2.12) is of course formally identical to the transformation into polar coordinates on the plane. Moreover, formulas (1.13) are also identical with the corresponding formulas of the plane problem and the problem of bending thin plates. Therefore the transformation (2.12) to the coordinates ρ and λ must lead to the same expressions for the shear reactions and moments as for the plane plate in polar coordinates. We obtain

$$S_\rho = -\frac{D}{h}\sqrt{\frac{12(m^2-1)}{m^2}}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\lambda^2}\right) \operatorname{Im} \sigma,$$

$$S_\lambda = -\frac{D}{h}\sqrt{\frac{12(m^2-1)}{m^2}}\frac{\partial^2}{\partial\rho^2} \operatorname{Im} \sigma, \tag{2.13}$$

$$S_{\lambda\rho} = \frac{D}{h}\sqrt{\frac{12(m^2-1)}{m^2}}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\lambda}\right) \operatorname{Im} \sigma,$$

$$G_\rho = -D\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{m\rho}\frac{\partial}{\partial\rho} + \frac{1}{m\rho^2}\frac{\partial^2}{\partial\lambda^2}\right) \mathrm{Re}\ \sigma,$$

$$G_\lambda = -D\left(\frac{\partial^2}{m\,\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\lambda^2}\right) \mathrm{Re}\ \sigma,$$

(2.13 cont.)
$$H_{\rho\lambda} = -D\frac{m-1}{m}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\lambda}\right) \mathrm{Re}\ \sigma,$$

$$Q_\rho^* = -D\left(\frac{\partial}{\partial\rho}\Delta + \frac{m-1}{m}\frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial^2}{\partial\lambda^2}\right) \mathrm{Re}\ \sigma,$$

$$Q_\lambda^* = -D\left(\frac{\partial}{\rho\,\partial\lambda}\Delta + \frac{m-1}{m}\frac{\partial^2}{\partial\rho^2}\frac{1}{\rho}\frac{\partial}{\partial\lambda}\right) \mathrm{Re}\ \sigma$$

$$\left(\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\lambda^2}\right).$$

We seek the solution of equation (2.7) in the form

(2.14) $$\psi_n = R_n \cos n\lambda \qquad \text{or} \qquad \psi_n = R_n \sin n\lambda,$$

where n is an integer. To determine $R_n(\rho)$ we obtain Bessel's differential equation

(2.15) $$R_n'' + \frac{1}{\rho} R_n' + \left(2i\beta^2 - \frac{n^2}{\rho^2}\right) R_n = 0.$$

We shall be interested in the solution of this equation which goes exponentially to zero for large values of the argument $\beta\rho$; this solution is given (as is known) by the first Hankel function of n-th order

(2.16) $$H_n^1(\beta\rho\sqrt{2i}) = \psi_n(\beta\rho) + i\chi_n(\beta\rho).$$

Returning to the solution (2.11) of the differential equation (1.14) we shall investigate it in the form

(2.17)
$$\sigma_n = [\Omega_1(\beta x) - 2i\Omega_3(\beta x)][\psi_n(\beta\rho) + i\chi_n(\beta\rho)]\cos n\lambda$$
$$= (\alpha_n + i\beta_n)\cos n\lambda,$$

when n is even, and in the form

$$\sigma_k = [\Omega_2(\beta x) - 2i\,\Omega_4(\beta x)][\psi_k(\beta\rho) + i\chi_k(\beta\rho)]\cos k\lambda$$
(2.18)
$$= (\alpha_k + i\beta_k)\cos k\lambda\,,$$

when k is odd.

For $\beta|x| \to \infty$ the functions $\Omega_i(\beta x)$ grow as $e^{\beta|x|}$. On the other hand, the asymptotic representation of the Hankel function has the form

$$H_n^{(1)}(\beta\rho\sqrt{2i}) \sim \frac{e^{-\beta\rho}}{\sqrt{\pi\beta\rho/\sqrt{2}}}\,[\cos(\beta\rho - \frac{4n+3}{8}\pi)$$
(2.19)
$$+ i\sin(\beta\rho - \frac{4n+3}{8}\pi)]\,.$$

Therefore σ_n and σ_k decrease for $\beta\rho \to \infty$ like

$$\frac{1}{\sqrt{\beta\rho}}\,e^{-\beta(\rho - |x|)}$$

and hence for λ, different from zero or π, go to zero exponentially and for $\lambda = 0$ and $\lambda = \pi$ not slower than $\frac{1}{\sqrt{\beta\rho}}$. It is also easily established that the solutions (2.17) - (2.18) are even with respect to x and λ.

For small values of the argument the functions α_i and β_i can be represented by the following series:

$$\alpha_o = \frac{1}{2} + \frac{1}{\pi}\beta^2\rho^2[(2 + \cos 2\lambda \log\frac{\gamma\beta\rho}{\sqrt{2}} - 1] + \dots,$$

$$\beta_o = \frac{2}{\pi}\log\frac{\gamma\beta\rho}{\sqrt{2}} - \frac{1}{4}\beta^2\rho^2(2 + \cos 2\lambda) + \dots,$$

$$\alpha_1 = \cos\lambda(\frac{2}{3\pi}\beta^2\rho^2 - \frac{1}{3\pi}\beta^2\rho^2\cos 2\lambda - \frac{2}{\pi}\beta^2\rho^2\log\frac{\gamma\beta\rho}{\sqrt{2}}) + \dots,$$

$$\beta_1 = \cos\lambda\,(\tfrac{1}{2}\,\beta^2\rho^2 - \tfrac{2}{\pi}) + \dots ,$$

$$\alpha_2 = -\frac{1}{\pi}\,\frac{2}{\beta^2\rho^2} - \frac{1}{2\pi}\,\beta^2\rho^2(\tfrac{1}{4} + \cos 2\lambda - \tfrac{2}{3}\cos 4\lambda + \log\frac{\gamma\beta\rho}{\sqrt{2}}) + \dots ,$$

$$\beta_2 = \frac{1}{\pi}\cos 2\lambda + \frac{1}{8}\,\beta^2\rho^2 + \dots ,$$

$$\alpha_3 = \cos\lambda[-\frac{8}{\pi\beta^2\rho^2} + \frac{2}{3\pi}\,\beta^2\rho^2(\tfrac{1}{4} - \tfrac{1}{2}\cos 2\lambda + \tfrac{2}{5}\cos^4\lambda)] + \dots ,$$

$$\beta_3 = -\frac{2}{3\pi}\cos\lambda\,(1 - 2\cos 2\lambda) + \dots ,$$

$$\alpha_4 = -\frac{1}{\pi}\,\frac{1}{\beta^2\rho^2}\,(10 + 6\cos 2\lambda) + \dots ,$$

$$\beta_4 = \frac{24}{\pi}\,\frac{1}{\beta^4\rho^4} + \dots .$$

Here γ stands for the Euler-Masceroni constant $\log\gamma = 0.5772\dots$

3. Let us investigate a cylindrical shell, with an opening bounded by the curve $\rho = \rho_o = \text{const.}$; if one unfolds this surface into a plane, it will have a circular cut-out with radius ρ_0. If the opening were not there the strained state in the wall of the cylinder would be given by the equations $S_1 = ph$ and $S_2 = qh$. We are interested in the deformation which this strained state is subjected to in the region close to the opening. This problem represents a straight generalization, for the case of a cylindrical surface, of Kirsch's problem for the distribution of stresses in a plane stress field with a circular opening. It is doubtful that one can obtain its exact solution. In the following we give an assymptotic solution, which is usable under the assumption that the parameter ρ_o^2/ah is small; this implies that in the following we assume that, althought a/h is large, the radius is sufficiently

small so that

$$\frac{\rho_o}{a} << \sqrt{\frac{h}{a}}. \tag{3.1}$$

We shall seek the complex stress function giving the deformation of the stressed state in the form

$$\begin{aligned}\sigma = (A + iB)\sigma_0 + (C + iD)\sigma_1 + (E + iF)\sigma_2 \\ + (H + iK)\sigma_3 + (L + iM)\sigma_4 + \ldots,\end{aligned} \tag{3.2}$$

where $\sigma_0, \ldots, \sigma_4$ are the solutions (2.17), (2.18) given above.

We must select the constants A, B, ... in such a way that certain conditions, given below, are satisfied on the contour of the opening. This turns out to be possible in the first approximation - at the cost of neglecting values which have the order of the square of the parameter ρ_o^2/ah given above. Let us set

$$\begin{aligned} A &= A_0 + A_1\beta^2 + \ldots, & B &= B_1\beta^2 + \ldots, \\ C &= C_0 + C_1\beta^2 + \ldots, & D &= D_1\beta^2 + \ldots, \\ E &= E_2\beta^4 + \ldots, & F &= F_1\beta^2 + F_2\beta^4 + \ldots, \\ H &= H_2\beta^4 + \ldots, & K &= K_3\beta^6 + \ldots, \\ L &= L_4\beta^8 + \ldots, & M &= M_3\beta^6 + \ldots, \end{aligned} \tag{3.3}$$

and using the expansion (2.20) represent σ as a series, only keeping terms of order β^2. We obtain for the real and imaginary parts of σ the expressions

$$\begin{aligned}\mathrm{Im}\,\sigma = \psi_o + \beta^2\psi_1 + \ldots = \frac{2A_o}{\pi}(\log\frac{\rho}{\rho_o} + \gamma') - \frac{C_o}{\pi}(1 + \cos 2\lambda) \\ - \frac{2F_1}{\pi}\ \frac{\cos 2}{\rho^2} + \beta^2[- \frac{1}{4} A_o\rho^2(2 + \cos 2\lambda) + \frac{C_o}{4}\rho^2(1 + \cos 2\lambda) \\ + \frac{2A_1}{\pi}(\log\frac{\rho}{\rho_o} + \gamma') + \frac{1}{2}B_1 - \frac{2F_2}{\pi}\frac{\cos 2\lambda}{\rho^2} - \frac{1}{\pi}C_1(1 + \cos 2\lambda)] + \ldots,\end{aligned} \tag{3.4}$$

$$\operatorname{Re}\sigma = w_o + \beta^2 w_1 + \dots = \frac{1}{2}A_o + \frac{\beta^2}{\pi}\left\{\frac{\pi A_1}{2} - 2B_1\gamma' + D_1 - \frac{F_1}{\pi}\right.$$

$$+ A_o(2\rho^2\log\frac{\rho}{\rho_o} + 2\rho^2\gamma' - \rho^2) - 2B_1\log\frac{\rho}{\rho_o} + C_o\rho^2(\frac{1}{4} - \log\frac{\rho}{\rho_o}$$

$$(3.5)\qquad - \gamma') + \cos 2\lambda[A_o\rho^2(\log\frac{\rho}{\rho_o} + \gamma') + C_o\rho^2(\frac{1}{6} - \log\frac{\rho}{\rho_o} - \gamma')$$

$$\left. + D_1 - \frac{2E_2}{\rho^2} - \frac{4H_2}{\rho^2}] - \cos 4\lambda(\frac{C_o}{12}\rho^2 + \frac{F_1}{2} + \frac{4H_2}{\rho^2} + \frac{24M_3}{\rho^4})\right\} .$$

Here we have introduced the notation

$$(3.6)\qquad \gamma' = \log\frac{\gamma\rho_o\beta}{\sqrt{2}} .$$

For $\beta = 0$, which corresponds to the case of a plane sheet with a circular opening, we obtain

$$(3.7)\qquad \psi_o = \operatorname{Im}\sigma = \frac{2A_o}{\pi}(\log\frac{\rho}{\rho_o} + \gamma') - \frac{C_o}{\pi}(1 + \cos 2\lambda) - \frac{2F_1}{\pi}\frac{\cos 2\lambda}{\rho^2} ,$$

$$(3.8)\qquad w_o = \operatorname{Re}\sigma = \frac{1}{2}A_o .$$

Let us note that σ = const. is a solution of (1.14). Therefore adding to (3.2) the constant

$$\sigma_* = -\frac{1}{2}A_o$$

we obtain $\operatorname{Re}\sigma = w = 0$ for $\beta = 0$, as is required.

We now determine the constants A_o, C_o, F_o from the condition that the stresses S_ρ and $S_{\lambda\rho}$ become zero on the contour of the opening; this, as in the Kirsch problem, leads to the conditions

$$S_\rho = \frac{1}{2}h(p + q) + \frac{1}{2}h(p - q)\cos 2\lambda$$

$$- \frac{D}{h}\sqrt{\frac{12(m^2 - 1)}{m^2}}\left[\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\lambda^2}\right)\psi_o\right]_{\rho = \rho_o} = 0 ,$$

(3.9)

(3.9 cont.) $$S_{\rho\lambda} = -\frac{1}{2}h(p - q)\sin 2\lambda + \frac{D}{h}\sqrt{\frac{12(m^2 - 1)}{m^2}}\left[\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\lambda}\psi_o\right]_{\rho = \rho_o} = 0 .$$

The first set of terms

$$\tfrac{1}{2}h(p + q)$$

here give the value of S_ρ and $S_{\rho\lambda}$ when no opening is present. The second set of terms in (3.9), determined from formulas (1.13), gives the deformation which a uniformly stressed plane plate undergoes when an opening is introduced. From (3.9) we find

$$A_o = \frac{\pi\rho_o^2}{4Eh}\sqrt{\frac{12(m^2 - 1)}{m^2}}(p + q) ,$$

(3.10) $$C_o = \frac{\pi\rho_o^2}{2\,Eh}\sqrt{\frac{12(m^2 - 1)}{m^2}}(p - q) ,$$

$$F_1 = -\frac{\pi\rho_o^4}{8Eh}\sqrt{\frac{12(m^2 - 1)}{m^2}}(p - q) .$$

We determine the constants A_1, F_2, C_1 now in such a way that the terms in the expressions for the stresses which contain the factor β^2 become zero for $\rho = \rho_o$; for this we must require

$$\left[\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\lambda^2}\right)\psi_1\right]_{\rho = \rho_o} = 0 ,$$

(3.11)

$$\left(\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\lambda}\psi_1\right)_{\rho = \rho_o} = 0$$

and obtain

$$A_1 = \frac{\pi\rho_o^2}{2}(A_o - \tfrac{1}{2}C_o), \quad C_1 = \frac{\pi\rho_o^2}{2}(C_o - A_o),$$

(3.12)

$$F_2 = -\frac{\pi\rho_o^4}{8}(C_o - A_o).$$

With these values of the constants the conditions

$$S_\rho = 0, \qquad S_{\rho\lambda} = 0 \tag{3.13}$$

will be satisfied on the contour of the opening up to values of order $\beta^2\rho_o^2$, inclusively, and for $\beta = 0$ the solution of the problem will coincide with the solution of the Kirsch problem.

Now we can determine in the region near the opening the stress S_λ:

$$S_\lambda = \tfrac{1}{2}h(p + q) - \tfrac{1}{2}h(p - q)\cos 2\lambda$$

(3.14)

$$-\frac{D}{h}\sqrt{\frac{12(m^2 - 1)}{m^2}}\frac{\partial^2\psi}{\partial\rho^2}.$$

A computation gives

$$\frac{1}{h}S_\lambda = \frac{1}{2}(p + q)(1 + \frac{\rho_o^2}{\rho^2}) - \frac{1}{2}(p - q)(1 + \frac{3\rho_o^4}{\rho^4})\cos 2\lambda$$

(3.15)

$$+\sqrt{\frac{3(m^2 - 1)}{m^2}}\frac{\pi\rho_o^2}{4ah}\left[(1 + \frac{\rho_o^2}{\rho^2})q - \frac{1}{4}(p - 3q)(1 + \frac{3\rho_o^4}{\rho^4})\cos 2\lambda\right] + \ldots .$$

Here instead of β^2 we used the value (2.3). In particular, on the contour of the opening we obtain

$$\left(\frac{S_\lambda}{h}\right)_{\rho = \rho_o} = p + q - 2(p - q)\cos 2\lambda$$

(3.16)

$$+\sqrt{\frac{3(m^2 - 1)}{m^2}}\frac{\pi\rho_o^2}{4ah}[2q - (p - 3q)\cos 2\lambda].$$

Let us look for example at the case of a cylinder with closed ends and subject to a constant interior pressure p_o : if no opening were present, we would have

$$(3.17) \qquad p = \frac{p_o a}{2h} \quad , \qquad q = \frac{p_o a}{h} \quad ,$$

and by (3.16) we obtain

$$(3.18) \qquad \left(\frac{S_\lambda}{h}\right)_{\rho = \rho_o} = \frac{p_o a}{h}\left[\frac{3}{2} + \cos 2\lambda + \sqrt{\frac{3(m^2 - 1)}{m^2}}\,\frac{\pi \rho_o^2}{4ah}\left(2 + \frac{5}{2}\cos 2\lambda\right)\right]$$

With zero curvature, the coefficient of the stress concentration at the opening is in this case equal to (2.5). If curvature is present this coefficient is multiplied by

$$(3.19) \qquad 1 + \frac{2.3}{3}\,\frac{\rho_o^2}{ah}$$

Let us consider the stresses which arise as a consequence of bending of the plate; these stresses become zero together with the curvature. Let us first of all require that on the contour of the opening the bending moment G_ρ becomes zero. Computing G_ρ by (2.13) and (3.5) we equate to zero the constant term and the coefficients of cos 2λ and cos 4λ ; we obtain three equations which together with the constants A_o, C_o, F_1, given above by (3.10), contain five unknowns

$$(3.20) \qquad B_1, \quad D_1, \quad E_2, \quad H_2, \quad M_3 ;$$

and we have two more constants at our disposal to satisfy the boundary conditions for the shear reaction Q_ρ^*. The expression for Q_ρ^* we must construct from (2.13) and (3.5). The two missing equations for our unknowns (3.20) we now obtain by setting the coefficients of cos 2λ and cos 4λ in the expression for Q_ρ^*,

for $\rho = \rho_o$, equal to zero. The constant term in this expression turns out to be, for $\rho = \rho_o$, equal to

$$(3.21) \qquad (Q_\rho^*)_{\rho = \rho_o} = - \frac{D\beta^2}{\pi} 8(A_o - \frac{C_o}{2}) = - \frac{\rho_o h}{2a} q$$

The resultant shear reactions are, therefore, equal to

$$2\pi\rho_o(Q_\rho^*)_{\rho = \rho_o} = - \pi\rho_o^2 \frac{qh}{a} = - \pi\rho_o^2 p_o \quad .$$

This could be expected since the shear forces must balance the resultant of the pressure on the surface of the opening.

It remains to determine the normal stresses on the contour of the opening produced by the bending moment $G_\lambda = \frac{1}{6}\sigma_\lambda' h^2$, where σ_λ' is the maximal value of the bending stresses (on the outer fibers). For this we must first determine the unknowns (3.20). Leaving out the computation, we give only the final results:

$$\sigma_\lambda' = - \frac{3\rho_o^2}{4ah} \frac{m+1}{m} [4q\gamma' + \frac{5}{2} q - \frac{1}{2} p - \cos 2\lambda \Big(\frac{p}{6} \frac{7m+5}{3m+1} - \frac{q}{6} \frac{13m+23}{3m+1} + \frac{2(m-1)}{3m+1} (3q - p)\gamma'\Big) + \frac{1}{2} (p - q) \frac{4m-3}{4m+3} \cos 4\lambda] \quad .$$

If, for instance, the stresses p and q are determined by (3.17), and if we take $1/m = 0.3$ and notice that then

$$\gamma' = \log \frac{\rho_o}{\sqrt{ah}} - 0.213$$

we obtain

$$\sigma_\lambda' = -\frac{q\rho_o^2}{ah}\left[3.9 \log \frac{\rho_o}{\sqrt{ah}} + 1.361\right.$$

(3.23)
$$\left. + \cos 2\lambda\left(0.996 - 1.035 \log \frac{\rho_o}{\sqrt{ah}}\right) - 0.154 \cos 4\lambda\right] .$$

The additional normal stresses for non-zero curvature, which are uniformly distributed along the thickness of the plate by (3.16), will be given by

(3.24)
$$\sigma_\lambda'' = 1.29 \frac{\rho_o^2 q}{ah} .$$

Values for these stresses and also values for the bending stresses along the contour of the opening for some values of the parameter $\rho_o^2/2a$ are given in the following table.

$\frac{\rho_o}{\sqrt{ah}}$	$-\sigma_\lambda'/q$	$-\sigma_\lambda'/q$	σ_λ''/q	σ_λ''/q
	$\lambda = 0$	$\lambda = \pi/2$	$\lambda = 0$	$\lambda = \pi/2$
0.5	- 0.055	0.786	1.458	- 0.162
0.4	0.065	0.683	0.932	- 0.104
0.3	0.112	0.514	0.523	- 0.058
0.2	0.095	0.307	0.233	- 0.009
0.1	0.044	0.111	0.058	- 0.006

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